Energy balance in feedback synchronization of chaotic systems

C. Sarasola
Department of Physics of Materials, University of the Basque Country, 20018 San Sebastian, Spain

F. J. Torrealdea, A. d'Anjou, A. Moujahid, and M. Graña
Department of Computer Science, University of the Basque Country, 20018 San Sebastian, Spain
(Received 8 July 2003; published 30 January 2004)

In this paper we present a method based on a generalized Hamiltonian formalism to associate to a chaotic system of known dynamics a function of the phase space variables with the characteristics of an energy. Using this formalism we have found energy functions for the Lorenz, Rössler, and Chua families of chaotic oscillators. We have theoretically analyzed the flow of energy in the process of synchronizing two chaotic systems via feedback coupling and used the previously found energy functions for computing the required energy to maintain a synchronized regime between systems of these families. We have calculated the flows of energy at different coupling strengths covering cases of both identical as well as nonidentical synchronization. The energy dissipated by the guided system seems to be sensitive to the transitions in the stability of its equilibrium points induced by the coupling.

DOI: 10.1103/PhysRevE.69.011606 PACS number(s): 05.45.Xt

I. INTRODUCTION

It is said that two different dynamical systems synchronize when they approach their behaviors as a consequence of their mutual interaction. The fact that it is possible to induce a synchronized regime between deterministic chaotic oscillators makes synchronization a phenomenon of significant interest in many areas of science and technology such as communication, electronics, optics, chemistry, and biology. Survey paper on different approaches in the synchronization and control of chaotic systems can be found in Ref. [1]. Usually the efficiency of a particular synchronization approach is only evaluated in terms of its ability to reach the established goal of proximity between the systems involved and very little is said about the cost, in terms of energy, of the process itself [2]. Nevertheless, some of the mechanisms described for the synchronization of nonidentical chaotic systems imply feedback interaction with coupling strengths going to infinity [3], and even the mechanism of complete replacement first reported in Ref. [4] is equivalent to a diffusive type of coupling with infinite gain [5], which might result in a demand of an unlimited amount of energy if a synchronized regime has to be reached and maintained. Much research on synchronization has been carried out working with theoretical systems for which it is not obvious how to define a measure of their behaviors in terms of energy and, consequently, how to establish the cost of their synchronization process. That is the case, for instance, of the very well-known chaotic systems of the Lorenz, Rössler, and Chua families. In this paper we develop a formal procedure to assign to a chaotic system of known dynamics a function of the variables of the phase space with the characteristics of an energy. Usually, it is the understanding of the energy and forces actuating on a system that permits to infer its kinetics. The approach in this work needs to be the opposite. Given the kinetics, we have to investigate what function of the phase space variables can be consistently thought of as a possible energy function for the system. This question finds a straightforward answer in Hamiltonian systems where the Hamiltonian function plays the role of the total energy of the system [6]. A system is Hamiltonian if it has the form $\dot{x}=M\nabla H(x)$; $x\in\mathbb{R}^{2n}$, where $H(x)$ denotes the Hamiltonian function; and $m=\begin{pmatrix}0 & I_d\\ -I_d & 0\end{pmatrix}$, where I_d denotes the identity matrix in $\mathbb{R}^n$. As M is a skew-symmetric matrix, $\dot{x}$ and $\nabla H(x)$ are always orthogonal. Consequently, the movement takes place at a constant value of the Hamiltonian (energy), that is, $H(x)$ is a first integral and the system is conservative. Nevertheless, dissipative chaotic systems are not conservative and neither can they be written in a Hamiltonian format nor have they a first integral energy function. Some kind of generalization of the Hamiltonian formalism, including dissipation, is then required in order to be able to analyze a dissipative chaotic system under this perspective. In Ref. [7] McLachlan *et al.* provide a general framework that encompasses both energy functions, in the conservative case, and Lyapunov functions, in the dissipative case, showing that they have a common formulation $\dot{x}=M(x)\boldsymbol{\nabla}H(x)$; $x\in\mathbb{R}^n$ and M is either a skew symmetric or a definite or semidefinite negative matrix. This generalization is compatible with the extended view of thinking of a Lyapunov function as a kind of generalized energy for dissipative systems. Nevertheless, dissipative chaotic systems do not fit into the McLaghlan *et al.* generalization as they have neither a first integral energy nor a Lyapunov function. A more general matrix M(x) is required to account for the kind of dissipation that takes place in dissipative chaotic systems. In Ref. [8] Bloch *et al.* express the general dynamics for systems with dissipation as sum of a skew symmetric Poisson bracket plus a symmetric bracket. This approach is also adopted in Refs. [9,10] using an ordinary matrix notation, and it is the one that we will be using in this paper when we refer to a generalized Hamiltonian formalism.

This generalized Hamiltonian approach has been used in problems of control [9,11], where typically positive definite quadratic forms are sought to play the role of energy. Unfortunately, as any positive definite quadratic form can always be forced to be a solution for the energy compatible with the generalized Hamiltonian formalism, independently of the system itself, the same trivial positive definite quadratic form has usually been assigned to different chaotic systems [9]. Nevertheless, assigning always the same type of energy function to every chaotic oscillator fails to uncover the individual traits of its particular dynamics. The generalized Hamiltonian approach requires additional hypotheses in order to be able to assign to each oscillator a particular energy function. These additional hypothesis can be established forcing a link between change in energy and change in phase space volume in the sense that both go together. Any energy variation cannot occur without a variation in the phase space volume and vice versa. We show in the paper that when this constraint is imposed to a particular chaotic oscillator as an additional condition to its generalized Hamiltonian representation it determines an energy function which is specific for the chaotic system and that is no longer, in general, a positive definite quadratic form. We would like to emphasize that this condition occurs naturally in ordinary physical systems, and that when this approach is applied to an ordinary physical system the energy obtained is the actual energy of the system.

Once the energies corresponding to two particular chaotic oscillators have been found, the flows of energy that take place when they synchronize their behaviors can be calculated. Many theoretical studies of chaos synchronization have been carried out coupling identical systems. In these cases, if feedback synchronization is used, identical synchronization is reached spontaneously at a given, usually small, value of the gain parameter k (coupling strength). Nevertheless, in most of the practical occurrences of synchronization the systems involved are not identical. They can be either nonidentical systems of the same family [12,13] or, even, systems of a completely different structure [14,15]. If nonidentical chaotic systems are forced to synchronize via feedback, synchronization does not spontaneously occur at a given value of the gain parameter k but, rather, it must be firmly enforced through the establishment of large values of the gain parameter. Identical synchronization between nonidentical systems is always a theoretical limit regime that would occur for coupling strengths going to infinity [3]. Nevertheless, the extent of proximity in the behavior of two systems that is going to be considered a synchronized regime will depend on the particular application considered and it will be, consequently, an experimental decision. As different degrees of synchronization can be required for different practical purposes, to know about the dependency of the flow of energy on the degree of synchronization can become an aspect of practical interest. Also, we show that maintaining the guided system in a synchronized regime requires an average nonzero flow of energy per unit time. This flow of energy should be provided, or absorbed, by the coupling device and compensate the interaction of the guided system with its environment through the dissipative components of its structure. This flow of energy, which can be assimilated to a dissipation process, turns out to be sensitive to some salient features of the bifurcation pattern of equilibriums induced by the coupling on the dynamical entity conformed by the guided system plus the coupling mechanism.

We develop in Sec. II the mathematical formalism to assign to any chaotic system of known structure a function of the variables of the phase space that could be formally considered as an energylike function of the system. Energy functions for the Lorenz, Rössler and Chua families of chaotic systems are found in Sec. III. Section IV. is devoted to study the energy balance in the feedback synchronization process of two chaotic oscillators. In Sec. V. the energy functions found in Sec. III are used to exhaustively compute the energy balance in the feedback synchronization of identical and different chaotic systems for different values of the gain parameter of the coupling term. Finally, a discussion of the results is presented in Sec. VI.

II. DETERMINATION OF THE ENERGY FUNCTION

Consider an autonomous dynamical system

$$\dot{x}=f(x), \tag{1}$$

where $x\in\mathbb{R}^n$ and $f:U\rightarrow\mathbb{R}^n$ is a smooth function with $U\subseteq\mathbb{R}^n$. These dynamical equations can be expressed in a generalized Hamiltonian form

$$\dot{x}=M(x)\nabla H, \tag{2}$$

where $M(x)$ is the local structure matrix and ∇H is the gradient vector of a smooth energy function $H(x)$. For Hamiltonian systems $M(x)$ is a skew-symmetric matrix which satisfies the Jacobian identity. For a generalized Hamiltonian system $M(x)$ is no longer skew symmetric but can be decomposed into the sum of a skew-symmetric matrix $J(x)$ and a symmetric matrix $R(x)$

$$\dot{x}=[J(x)+R(x)]\nabla H. \tag{3}$$

The time derivative of the energy along a trajectory is then

$$\dot{H}=\nabla H^T[J(x)+R(x)]\nabla H=\nabla H^T R(x)\nabla H \tag{4}$$

as for the skew-symmetric matrix $J(x)$,

$$\nabla H^T J(x)\nabla H=0. \tag{5}$$

In many physical problems the local structure matrix $M(x)$ and the energy function $H(x)$ of the dynamical system are known, and then the energy change in time is easily evaluated by Eq. (4). In our case we only know the vector field given by Eq. (1) and we do not know either the energy function of the system or its structure matrix. The problem is then to associate to the dynamical system an energy function and a local structure matrix compatible with its dynamics, that is, in the form of Eq. (2). This association is not unequivocal, and to use as an energy function the trivial quadratic positive definite function of the state variables [9] is frequent. In doing so, the procedure assigns to every dynamical system the same type of energy function and fails to uncover the particular characteristics of its dynamics. We adopt here a different approach. As Eq. (2) does not uniquely determine matrix $M(x)$ and energy H, additional hypotheses are required in order to use that formalism to assign a specific energy function to the system given by Eq. (1). In ordinary physical systems any energy variation that occurs as a consequence of their dynamics always takes place together with a volume change in phase space. In what follows we show that if we impose this natural condition to the energy function to be associated to the dissipative chaotic oscillator given by Eq. (1), this energy function becomes unique.

According to Liouville's theorem, the volume rate of change in phase space associated to the vector velocity field f is related with the divergence of that field by $dV(t)/dt = \int_{A(t)} \mathrm{div}\, f(x) dx = \int_{A(t)} \Sigma_{i=1}^{n} \partial f_i / \partial x_i dx$, where A is a bounded set in the phase space $\mathbb{R}^n$ and V its volume. If we could isolate unequivocally from the vector field f the component that contributes to its divergence we would be able to determine the energy associated to that vector field, imposing the condition that any temporal variation of the energy along a trajectory of the system occurs exclusively due to the presence of that component.

Helmholtz's theorem [16] guarantees that we can decompose a vector field f into the sum of one divergence-free vector f_c that accounts for the whole rotational tensor of f plus one gradient vector field f_d that carries its whole divergence.

$$f(x) = f_c(x) + f_d(x). \tag{6}$$

In practice, we can construct the vector field f_d taking all the terms of f that contribute to its divergence and only those terms. The rest of the terms of the vector field f form f_c.

The decomposition given by Eq. (6) can be used to determine the energy associated with the system $\dot{x} = f(x)$ imposing the condition that any change of the energy along a trajectory of the system occurs exclusively due to the contribution of the term f_d.

If we impose in Eq. (4) the condition

$$R(x) \nabla H = f_d(x), \tag{7}$$

we have

$$\dot{H} = \nabla H^T f_d(x). \tag{8}$$

That is, the energy is dissipated, passively or actively, due to the divergent component of the velocity vector field and can be thought of as the work per unit time of the energy gradient along this velocity component according to Eq. (8).

To determine the energy function H that fulfills this requirement it is sufficient to realize that if Eq. (7) holds, then Eq. (3) can be rewritten as

$$\dot{x} = [J(x) + R(x)] \nabla H = J(x) \nabla H + f_d(x), \tag{9}$$

and, consequently,

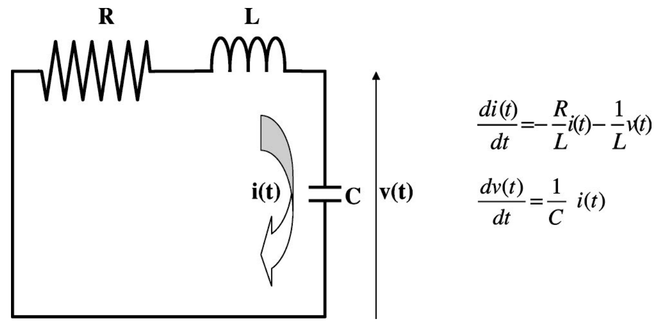

FIG. 1. An *RLC* electrical network.

$$J(x) \nabla H = f_c(x). \tag{10}$$

On the other hand, for $J(x)$ to be a skew-symmetric matrix,

$$\nabla H^T J(x) \nabla H = 0 \tag{11}$$

or

$$\nabla H^T f_c(x) = 0, \tag{12}$$

which defines for each dynamical system a partial differential equation from which the energy function $H(x)$ can be calculated.

Once the energy function $H(x)$ is known, the system can be easily rewritten in a $\dot{x} = [J(x) + R(x)] \nabla H(x)$ formulation to make explicit the corresponding skew-symmetric $J(x)$ and symmetric $R(x)$ matrices. We would like to point out that whereas the energy function is unequivocally determined by the velocity vector field f, the matrices $J(x)$ and $R(x)$ are not, which simply shows the fact that different formulations can be compatible with the same dynamics.

III. ENERGY FUNCTIONS FOR SOME WELL-KNOWN CHAOTIC OSCILLATORS

In this section we apply the above procedure to assign energy functions to three well known families of chaotic oscillators, Lorenz, Rössler, and Chua. In these three cases, as $x \in \mathbb{R}^3$, we will use the standard notation x, y, z for the phase space variables. First, we would like to illustrate our point finding an energy function for an ordinary dissipative electrical oscillator. This circuit will also be used as an introductory example for the analysis of the balance of energy during the synchronization process.

Consider the series *RLC* electrical network of Fig. 1, where R is the resistance of the resistor, L the inductance of the coil, and C is the capacity of the capacitor. The state variables are the current i through the circuit and the voltage difference v between the terminals of the capacitor. This electrical circuit is modeled by the equations

$$\dot{i} = -\frac{R}{L} i - \frac{1}{L} v,$$

$$\dot{v} = \frac{1}{C} i. \tag{13}$$

If we identify in the velocity vector field the part responsible for the divergence of the field, f_d, and the part that does not contribute to it, f_c, we have

$$f_{\mathrm{c}}=\begin{pmatrix} -\frac{1}{L}v \\ \frac{1}{C}i \end{pmatrix} \quad \text{and} \quad f_{\mathrm{d}}=\begin{pmatrix} -\frac{R}{L}i \\ 0 \end{pmatrix}. \tag{14}$$

Then, according to Eq. (12), the energy H associated with the circuit will satisfy the partial differential equation

$$-\frac{v}{L}\frac{\partial H}{\partial i}+\frac{i}{C}\frac{\partial H}{\partial v}=0, \tag{15}$$

which is satisfied by the quadratic form

$$H=\tfrac{1}{2}(Li^2+Cv^2), \tag{16}$$

which corresponds to the energy usually associated with the electrical circuit, as sum of the potential energy in the coil plus the energy accumulated in the capacitor. Note that the component f_c of the vector field is conservative with respect to H as it does not contribute to the change of the energy H along a trajectory of the system.

We can also find, according to Eq. (8), the rate of change of this energy along a trajectory of the system.

$$\dot{H}=\boldsymbol{\nabla} H^T f_d(x)=(Li,Cv)\begin{pmatrix} -\frac{R}{L}i \\ 0 \end{pmatrix}=-Ri^2=-v_R i, \tag{17}$$

where v_R is the terminal voltage in the resistor. Thus, we can see that the described procedure determines a dissipation process that occurs in the correct place, the resistor, and at the appropriate rate, $-v_R\, i$.

A. Lorenz

In this section we look for a function of the phase space variables that could be consistently considered as an energy function for the Lorenz family of systems. Let $\dot{x}=f(x)$ be the following Lorenz system:

$$\begin{aligned}\dot{x}&=\sigma y-\sigma x,\\ \dot{y}&=\rho x-y-xz,\\ \dot{z}&=xy-\beta z.\end{aligned} \tag{18}$$

To find a decomposition of the velocity vector field f of the type described by Eq. (6), we first investigate which terms in each component of the velocity field contribute to its divergence. These terms, and only these, define the vector field f_d. The remaining terms form the vector field f_c. We identify the following vector fields:

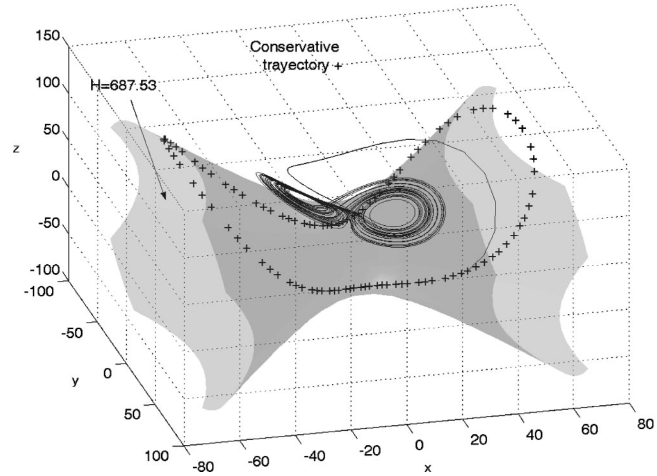

FIG. 2. Isosurface corresponding to a constant energy value $H=687{,}53$ for a Lorenz system with parameters (16, 45.92, 4). Energy is in arbitrary units.

$$f_c=\begin{pmatrix} \sigma y \\ \rho x-xz \\ xy \end{pmatrix} \quad \text{and} \quad f_d=\begin{pmatrix} -\sigma x \\ -y \\ -\beta z \end{pmatrix}. \tag{19}$$

As it can be observed f_d is a gradient vector that carries the whole divergence of the vector field f and f_c is a divergence free-vector that takes account of its whole rotor. So, the decomposition $f=f_c+f_d$ of the velocity vector field of the Lorenz system given by Eq. (18) satisfies the conditions of Helmholtz's theorem.

Consequently, according to Eq. (12), the energy function $H(x,y,z)$ will obey the following partial differential equation:

$$\sigma y\frac{\partial H}{\partial x}+(\rho x-xz)\frac{\partial H}{\partial y}+xy\frac{\partial H}{\partial z}=0, \tag{20}$$

one solution being the nondefinite quadratic form

$$H=\frac{1}{2}\left(-\frac{\rho}{\sigma}x^2+y^2+z^2\right). \tag{21}$$

The derivative of this energy along a trajectory is according to Eq. (8):

$$\dot{H}=\rho x^2-y^2-\beta z^2. \tag{22}$$

Figure 2 shows the isosurface of constant energy $H=687.53$ (arbitrary units) for a Lorenz system with parameters $\sigma=16$, $\rho=45.92$, $\beta=4$. The location of the actual Lorenz attractor and a trajectory corresponding to its conservative component $\dot{x}=f_c(x)$ can also be seen.

Once the energy function H is known, the Lorenz system can be easily rewritten, according to Eq. (3), as sum of a skew-symmetric matrix $J(x,y,z)$ and a symmetric matrix $R(x,y,z)$. Note that matrices $J(x,y,z)$ and $R(x,y,z)$ are not unique. The following is an example of decomposition of the Lorenz system where the symmetric matrix takes a diagonal form

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix}=\left[\begin{pmatrix}0 & \sigma & 0\\ -\sigma & 0 & -x\\ 0 & x & 0\end{pmatrix}+\begin{pmatrix}\frac{\sigma^2}{\rho} & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -\beta\end{pmatrix}\right]\times\begin{pmatrix}\frac{-\rho}{\sigma}x\\ y\\ z\end{pmatrix}. \tag{23}$$

B. Rössler

If we apply the same procedure to the Rössler system

$$\dot{x}=-y-z,$$
$$\dot{y}=x+ay, \tag{24}$$
$$\dot{z}=b+(x-c)z,$$

we obtain

$$f_c=\begin{pmatrix}-y-z\\ x\\ b\end{pmatrix} \quad \text{and} \quad f_d=\begin{pmatrix}0\\ ay\\ (x-c)z\end{pmatrix}.$$

This time the vector field f_d is not a gradient and, consequently, this decomposition does not satisfy the conditions of Helmholtz's theorem. That is, although f_d carries the whole divergence of the velocity field f, it still retains part of its rotor. A quick inspection shows that the addition to f_d of the divergence-free vector $(1/2z^2,0,0)^T$ compensates its rotor while keeping its divergence unchanged. Thus Helmholtz's decomposition of the velocity field f of the Rössler system will be $f=f_c+f_d$, with

$$f_c=\begin{pmatrix}-y-z-1/2\,z^2\\ x\\ b\end{pmatrix} \quad \text{and} \quad f_d=\begin{pmatrix}1/2z^2\\ ay\\ (x-c)z\end{pmatrix}, \tag{25}$$

where f_c carries the rotor of f and f_d its divergence.

Consequently, according to Eq. (12), the energy function $H(x,y,z)$ will obey the following partial differential equation:

$$-(y+z+1/2\,z^2)\frac{\partial H}{\partial x}+x\frac{\partial H}{\partial y}+b\frac{\partial H}{\partial z}=0, \tag{26}$$

which has the solution

$$H=\tfrac{1}{2}\{[x+b(z+1)]^2+(y+z^2/2+z-b^2)^2\}. \tag{27}$$

The derivative of this energy along a trajectory will be given according to Eq. (8) by

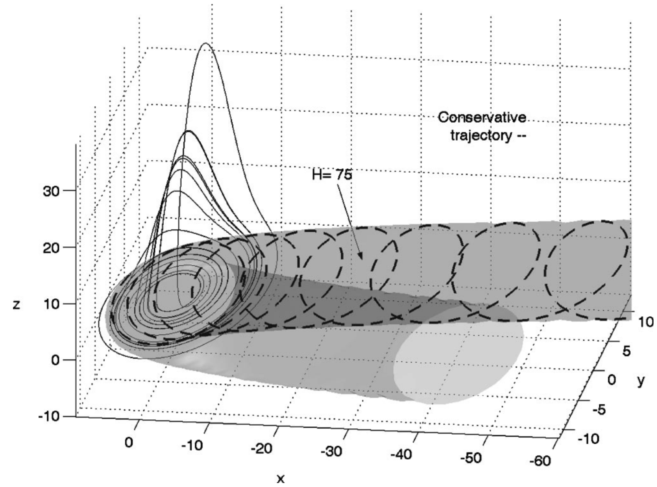

FIG. 3. Isosurface corresponding to a constant energy value $H=75$ for a Rössler system with parameters 0.2, 0.2, 5.7. Energy is in arbitrary units.

$$\dot{H}=\nabla H^T\begin{pmatrix}1/2z^2\\ ay\\ (x-c)z\end{pmatrix}. \tag{28}$$

In Fig. 3 a Rössler attractor, $a=0.2$, $b=0.2$, $c=5.7$, along with the isosurface corresponding to energy $H=75$, in arbitrary units of the phase space, is shown. The trajectory on the surface corresponds to the conservative component $\dot{x}=f_c(x)$ of the Rössler dynamics in the particular decomposition performed in this work. Note that the energy function H is not, strictly speaking, an exclusive characteristic of the Rössler system but rather of any system with the same conservative component f_c. On the other hand, the derivative of the energy along a trajectory, $\dot{H}$, is strictly linked to the Rössler dynamics as it is a direct consequence of both dynamic components, the conservative f_c and the dissipative f_d. The same consideration obviously applies to the other families of chaotic systems that we are considering in this work.

C. Chua

For the case of a continuous Chua system given by the equations

$$\dot{x}=\alpha y-\alpha x^3-\alpha c x,$$
$$\dot{y}=x+z-y, \tag{29}$$
$$\dot{z}=-\beta y,$$

the following vector fields

$$f_c=\begin{pmatrix}\alpha y\\ x+z\\ -\beta y\end{pmatrix} \quad \text{and} \quad f_d=\begin{pmatrix}-\alpha x^3-\alpha c x\\ -y\\ 0\end{pmatrix} \tag{30}$$

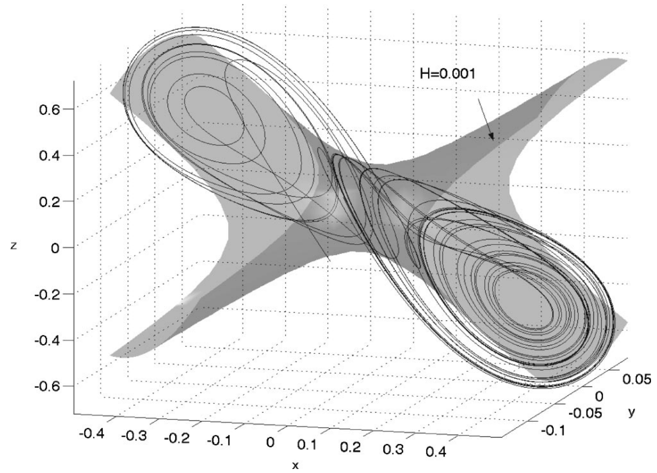

FIG. 4. Isosurface corresponding to a constant energy value $H=0.001$ for a Chua system with parameters (10, 16, −0.143). Energy is in arbitrary units.

are, respectively, divergence-free and irrotational and, consequently, we have the following law for the energy function $H(x,y,z)$,

$$\alpha y \frac{\partial H}{\partial x} + (x+z)\frac{\partial H}{\partial y} - \beta y \frac{\partial H}{\partial z} = 0, \tag{31}$$

which is satisfied by the nondefinite quadratic form

$$H = \frac{1}{2}\left(-\frac{1}{\alpha}x^2 + y^2 + \frac{1}{\beta}z^2\right), \tag{32}$$

with time variation along a trajectory

$$\dot{H} = x^4 + cx^2 - y^2. \tag{33}$$

In Fig. 4 a Chua attractor, $\alpha=10$, $\beta=16$, $c=-0.143$, along with the isosurface $H=0.001$, can be seen.

IV. FEEDBACK SYNCHRONIZATION ENERGY BALANCE

In the preceding section we have assigned different energy functions to different chaotic oscillators. The existence of a function of the phase space variables that measures the energy of a particular state of a given chaotic system permits evaluation of the energy exchange of the system with its environment when it moves along a particular trajectory. The energy derivative given by Eq. (8) measures the energy exchange of system $\dot{x}=f(x)$. It can be thought of as a dissipation process that takes place in the divergent constituents of the system. The energy derivative given by Eq. (8) can be either positive or negative, and, consequently, the exchange of energy that the system maintains with its environment should be understood as being sometimes an active and sometimes a passive, dissipation process. An autonomous chaotic oscillator initially located outside its natural attractor will lose, or gain, energy in its movement towards its natural oscillatory region of phase space where its net average energy variation will be zero. This is so because on the attractor the trajectory will repeatedly return to arbitrarily close states in phase space and consequently to arbitrarily close energy values. Thus, on the attractor the time average of the energy rate given by Eq. (8) will be zero,

$$\langle [\nabla H^f(x)]^T f_d(x) \rangle = 0, \tag{34}$$

where the brackets represent averaging in time and H^f denotes the energy function of system $\dot{x}=f(x)$.

So far, system $\dot{x}=f(x)$ has been considered as an autonomous dynamical system. In this section we intend to evaluate the energy balance that takes place when a system is forced to synchronize another guiding system. A chaotic oscillator $\dot{x}=f(x)$ can be forced to synchronize a different guiding chaotic system $\dot{y}=g(y)$ via feedback coupling according to the equations

$$\dot{y} = g(y),$$

$$\dot{x}_k = f(x_k) + K(y - x_k), \tag{35}$$

where $x,y \in \mathbb{R}^n$, $f,g:\mathbb{R}^n \to \mathbb{R}^n$ are smooth functions, K is the $n\times n$ diagonal matrix with diagonal entries $k>0$, a gain parameter that measures the strength of the coupling, and $x_k(t)$ indicates the state of the guided system when the gain parameter is set to k. Note that $K(y-x_k)$ is the coupling interface required in order to be physically able to implement the coupling of both systems $\dot{x}=f(x)$ and $\dot{y}=g(y)$.

If the oscillatory system $f(x_k)$ is maintained in a forced regime outside its natural attractor, Eq. (8) will produce a net nonzero average dissipation rate. Nevertheless, considering the whole entity $f(x_k)+K(y-x_k)$, the trajectory $x_k(t)$ remains, for every value of k, confined to an attractive region of phase space [17] and the net average energy variation corresponding to system $f(x_k)+K(y-x_k)$ will also be zero. That is,

$$\langle [\nabla H^f(x_k)]^T [f_d(x_k) + K(y-x_k)] \rangle = 0, \tag{36}$$

from which,

$$\langle [\nabla H^f(x_k)]^T K(y-x_k) \rangle = -\langle [\nabla H^f(x_k)]^T f_d(x_k) \rangle. \tag{37}$$

According to Eq. (37), the coupling device provides the flow of energy needed to compensate the energy exchange of system $f(x_k)$ with its environment. Thus, the amount of energy per unit time $P(k)$ that is necessary to provide the guided system with in order to maintain the degree of synchronization attained with a coupling of gain parameter k and, consequently, forced to follow an unnatural trajectory $x_k(t)$, will be

$$P(k) = -\langle [\nabla H^f(x_k)]^T f_d(x_k) \rangle. \tag{38}$$

This energy can be considered as the cost of maintaining that particular level of synchronization.

The degree of synchronization reached, measured in terms of the error vector $e=x_k-y$, depends on the magnitude of the gain parameter k. The norm of the synchronization error can be made arbitrarily small, as long as a sufficiently large gain k is implemented. To find the cost of maintaining a

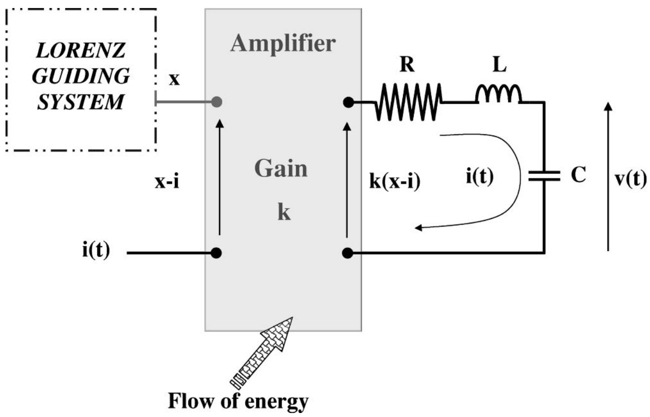

FIG. 5. An *RLC* oscillatory electrical circuit coupled via feedback to a chaotic Lorenz system.

regime of complete synchronization we can substitute $y(t)$ for $x_k(t)$ in Eq. (38) to obtain

$$\lim_{k\to\infty} P(k) = -\langle [\nabla H^f(y)]^T f_d(y)\rangle. \tag{39}$$

As variable $y(t)$ is confined to move in the attractor of the guiding system, Eq. (39) shows that the cost, or power needed to maintain both systems completely synchronized, remains bounded in spite of the fact that its attainment might imply arbitrarily large values of the gain parameter k.

V. COMPUTED SYNCHRONIZATION ENERGY

In this section we present computational results concerning the energy balance of the synchronization process of some chaotic systems in a wide range of values of the gain parameter k. First we introduce the subject with an illustrative example where the *RLC* circuit studied in Sec. III tries to synchronize its behavior to a chaotic Lorenz signal. Second, we study in great detail the transition towards identical synchronization of two coupled identical Lorenz systems and also of two coupled identical Chua systems. Finally, synchronizing different chaotic systems is studied in the cases of a Chua guided Rössler system and a Chua guided Lorenz system.

Let us consider the *RLC* circuit of Fig. 5 coupled via feedback to a scalar signal corresponding to the variable x of a Lorenz system with parameters $\sigma=16$, $\rho=45.92$, $\beta=4$. The guiding signal is chaotic for these particular values of the parameters. As it can be appreciated in Fig. 5, the coupling requires an electronic amplifier to set the appropriate voltage that physically implements the interaction term $k(x-i)$. The complete set of equations that models the synchronization process is then

$$\dot{i}_k = -\frac{R}{L} i_k - \frac{1}{L} v_k + k(x-i_k),$$

$$\dot{v}_k = \frac{1}{C} i_k. \tag{40}$$

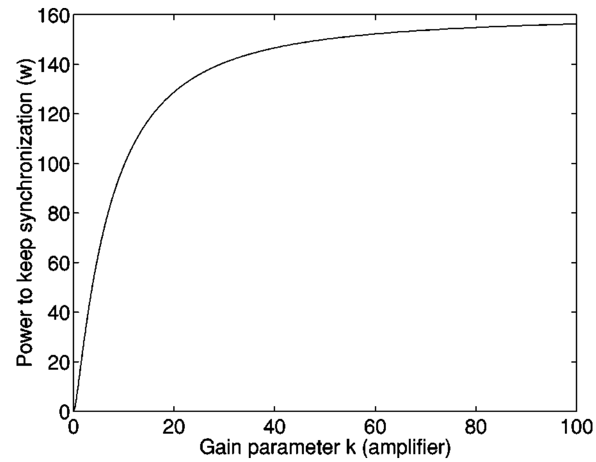

FIG. 6. Power required to synchronize an oscillatory electrical circuit with $R=1\,\Omega$, $L=1$ H, and $C=1$ F, to a Lorenz signal at different gain values. Power is in watts.

The power $P(k)$ that is necessary to provide the guided system with in order to maintain the synchronized regime attained by the circuit at a gain k will be, according to Eqs. (38) and (17),

$$P(k) = -\left\langle (Li_k, Cv_k)\begin{pmatrix} -\frac{R}{L} i_k \\ 0 \end{pmatrix}\right\rangle = \langle R i_k^2\rangle. \tag{41}$$

Note that this power supply is delivered to the system, via the electronic amplifier, from an external energy source. Figure 6 shows computed results of the power needed to maintain the synchronized regime of the oscillatory circuit with $R=1\,\Omega$, $L=1$ H, and $C=1$ F, at different values of the gain parameter k. As it can be appreciated the required power to maintain complete synchronization of the current i tends towards a limit value of about 160 watts at very large values of the gain parameter k.

A. Identical synchronization

Many works on synchronization of chaotic systems are concerned with synchronizing systems with the same structure and the same parameter values. In this case a synchronized regime of zero error, identical synchronization, is usually obtained when the gain parameter k of the coupling is set beyond a certain value, usually small. In the two examples that follow the setting up of the coupling situates the guided system in a dissipative regime that can only be maintained with a continuous provision of energy through the coupling device. The energy dissipated per unit time increases with k until, abruptly, an identical synchronization regime is reached and the cost of maintaining that synchronized regime becomes zero.

1. A Lorenz system guiding another identical Lorenz system

We have chosen two identical Lorenz systems with parameters $\sigma=16$, $\rho=45.92$, $\beta=4$, coupled via feedback coupling in the way described by Eq. (35). At these parameter

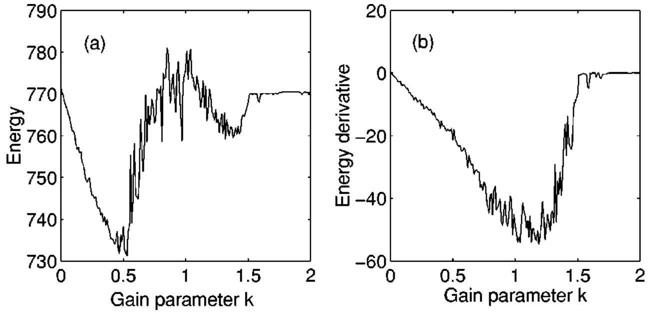

FIG. 7. A Lorenz system with parameters $\sigma=16$, $\rho=45.92$, $\beta=4$ guiding another identical Lorenz system. In part (a), average energy of the guided system at different values of the gain parameter k. In (b), average energy dissipated per unit time by the guided system at different values of the gain parameter k. Energy is in arbitrary units.

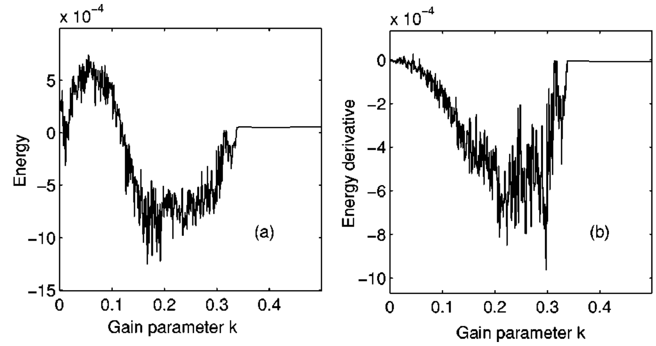

FIG. 8. A Chua system with parameters $\alpha=10$, $c=-0.143$, $\beta=16$ guiding another identical Chua system. In part (a), average energy of the guided system at different values of the gain parameter k. In (b), average energy dissipated per unit time by the guided system at different values of the gain parameter k. Energy is in arbitrary units.

values the Lorenz system operates in a chaotic regime. The gain parameter k has been varied smoothly ranging from $k=0$ to $k=2$. For each value of k the energy $H(x_k)$ given by Eq. (21) and its time derivative $\dot{H}(x_k)$ given by Eq. (22) have been averaged along a trajectory of the coupled system long enough as to be considered averaged on the attractor. The results are shown in Figs. 7(a) and 7(b), respectively.

As it can be seen, the coupling interface makes the guided system attractor move through phase space regions of different energy following a waving average energy pattern. For all these values of the gain k the systems are not yet synchronized. Identical synchronization occurs at values of the gain parameter in the neighborhood of $k=1.6$ where the average energy of the guided system returns to its original level. As can be seen in Fig. 7(b) the derivative of the energy of the original guided system, in the sense described by Eq. (37), along a trajectory x_k of the coupled system follows a different pattern. As soon as the coupling is connected, the average energy derivative of the guided system becomes negative, that is, it starts to dissipate on average an energy that the coupling device will have to provide in order to maintain the forced regime. The required energy increases linearly, with two different slopes, with the gain parameter k, until the onset of the identical synchronization stage. At values of k in the neighborhood of $k=1.2$ some structural change must happen that would permit the guided system to reach very quickly identical synchronization at the already mentioned value of the gain $k=1.6$ with no energy consumption at all. We conjecture that the onset of this identical synchronization stage is linked to the transition to stable spiral at $k=0.77$ of the previously two unstable spirals equilibrium points of the perturbed Lorenz system. We will elaborate on this idea in the following section.

2. A Chua system guides another identical Chua system

As a different example of identical synchronization we have performed computational experiments with two identical Chua continuous systems with parameters $\alpha=10$, $c=-0.143$, $\beta=16$ linked together with the same type of feedback coupling described before. This set of parameter values makes the Chua system itself maintain a chaotic behavior. The results are shown in Fig. 8.

As in the previous case the average energy varies with parameter k in a waving pattern that abruptly ends at the moment synchronization is reached. The Chua attractor is a small region around the origin of phase space, that is why the absolute values of the exchanged energy are in this case much smaller than in the previous case. As long as the gain is not large enough as to force identical synchronization, the system is dissipating energy on average, and, consequently, to maintain that regime at any given value of k requires a provision of energy. For values of k around $k=0.35$ identical synchronization is reached, at a zero energy maintenance cost. This case and the previously described Lorenz guiding Lorenz case exhibit a qualitatively similar pattern of energy variation which could probably be considered representative of the feedback identical synchronization behavior.

3. Sensitivity to perturbations

Real systems are unlikely to be identical as some parameter mismatch can always be expected. If two systems are not exactly the same, identical synchronization at zero cost will not spontaneously occur. Nevertheless, a regime of nearly complete synchronization can be forced, at large values of the coupling parameter k, whose maintenance will demand a limited amount of energy per unit time $P=\lim_{k\to\infty}P(k)$, given by Eq. (39).

Equation (39) can be used to evaluate the sensitivity of this limit flow of energy P to the parameter mismatch. Let us suppose that in the previously studied Lorenz guiding Lorenz case, x, y, and z are the variables of the guiding Lorenz system with parameters σ, ρ, and β, and the mismatch in the parameters of the intended identical response Lorenz system are respectively, $\delta\sigma$, $\delta\rho$, and $\delta\beta$. According to Eqs. (39) and (22), the energy per unit time, δP, required to maintain a synchronized regime will be

$$\delta P=\langle(\rho+\delta\rho)x^2-y^2-(\beta+\delta\beta)z^2\rangle. \tag{42}$$

Taking into consideration that the average energy dissipated by the response system is zero when its parameters are identical to the corresponding parameters in the guiding system, we have

$$\delta P = \delta\rho\langle x^2\rangle - \delta\beta\langle z^2\rangle. \tag{43}$$

Equation (43) shows that the synchronization energy due to the parameter mismatch is not sensitive to parameter σ, and that its sensitivity to parameter ρ can be found by computing the average value of the square of the guiding variable x, while its sensitivity to parameter β is given by minus the average value of the square of the guiding variable z. Note that these results refer to the theoretical limit case of a gain parameter k going to infinity.

The presence of noise will also prevent two theoretically identical systems from reaching a regime of identical synchronization at zero cost. Equation (38) can be used to compute the cost at different values of the gain parameter k.

B. Synchronizing different systems

When the systems to be synchronized are different the mechanisms governing the dynamics of the synchronization are likely to be more complicated than in the case of identical systems. A possible approach to its understanding is to contemplate the guided system and its coupling device in the second of Eqs. (35) as $\dot{x}_k = f(x_k) - Kx_k + Ky$ which shows the original structure of the guided system $f(x_k)$ perturbed by the coupling device to $f(x_k) - Kx_k$ plus an exogenous input Ky. For different values of the gain parameter k, the whole lot of limit sets of the perturbed guided system experiences a pattern of bifurcations which can be relevant for the dynamics of the coupling system, specially for low values of k and weak leading signals. In particular, when studying the energy balance of the synchronization at different values of the gain parameter, the bifurcation pattern of equilibrium points could determine the behavior of the coupling at low values of k, while what is going to happen for larger values of the gain parameter could be more dependent on the characteristics of the master system. That is, roughly speaking, the traits of the coupled regime reached at low values of the gain parameter, weak couplings, are likely to be characteristic of the slave system itself and relatively independent of the master system, while for strong couplings the guiding system becomes dominant. Sometimes very weak couplings can be able to produce stable periodic orbits or stable points in the perturbed system which induce the onset of synchronization phenomena such us phase synchronization or general synchronization [1] or even regimes of nearly complete synchronization at low values of energy exchange.

In what follows a Chua guided Lorenz system and a Chua guided Rössler system are studied.

1. A Chua continaous system guides a Rössler system

We study here the case of synchronizing two different chaotic systems where a Chua continuous system with parameters $\alpha=10$, $c=-0.143$, $\beta=16$ guides a Rössler system with parameters $a=0.2$, $b=0.2$, $c=5.7$. The parameter values have been chosen to guarantee a chaotic free behavior of the drive and response systems.

The intrinsic dynamics of the autonomous family of Rössler systems perturbed by the coupling , that is, $\dot{x}_k = f(x_k) - Kx_k$, where $f(x_k)$ stands for the particular Rössler guided system studied, exhibits a bifurcation diagram of equilibrium points as a function of the coupling parameter k that can explain some salient features of the computed energy. For $k=0$, the unperturbed initial Rössler system has a weakly unstable spiral saddle point very near the origin whose stability condition is very soon altered as a function of the parameter k. For $k=0.04$ the perturbed system $\dot{x}_k = f(x_k) - Kx_k$ has a stable limit cycle that collapses into a stable spiral node, by $k=0.098$. This bifurcation scheme can explain the fine structure of the observed energy at very low values of the parameter k. In fact, the pattern of Lyapunov exponents of the response system at these values of the gain parameter k is complicated but at $k=0.098$ the largest Lyapunov exponent becomes negative and generalized synchronization [18] occurs due to the destruction of the specific dynamics of the $\dot{x}_k = f(x_k) - Kx_k$ perturbed system. The onset of this synchronized regime is detected by a sudden decline of the energy exchanged per unit time as it can be seen in Fig. 9(a). For larger values of the gain parameter the Rössler guided system is progressively dragged towards the basin of attraction of the guiding Chua system to a regime of complete synchronization that takes place at the expense of a net income flow of energy, of about 0.8 arbitrary units per second, required to maintain its dissipative condition. In Fig. 9(b) the average derivative of the Rössler system energy is presented versus a high range of values of the gain parameter k.

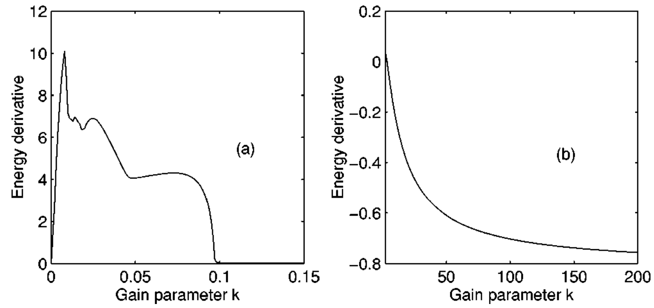

FIG. 9. A Chua system with parameters $\alpha=10$, $c=-0.143$, $\beta=16$ guiding a Rössler system with parameters $a=0.2$, $b=0.2$, $c=5.7$. In part (a), average energy dissipated per unit time by the guided system at small values of the gain parameter k. In part (b) a larger range of values of the gain parameters. Energy in arbitrary units.

2. A Chua continuous system guiding a Lorenz system

In this case a Chua continuous system with parameters $\alpha=10$, $c=-0.143$, $\beta=16$ guides a Lorenz system with parameters $\sigma=16$, $\rho=45.92$, $\beta=4$. With those values of the parameters the free behavior of the drive and response systems is chaotic. Due to the relatively low energy values corresponding to the states of the attractor of the guiding Chua system, around $H=0.001$ as it can be appreciated in Fig. 4, the variation of the energy balance of the synchronization as a function of the gain parameter k depends on the particular structure of the bifurcations of the equilibriums of the perturbed family of guided systems in the sense expressed in the preceding section.

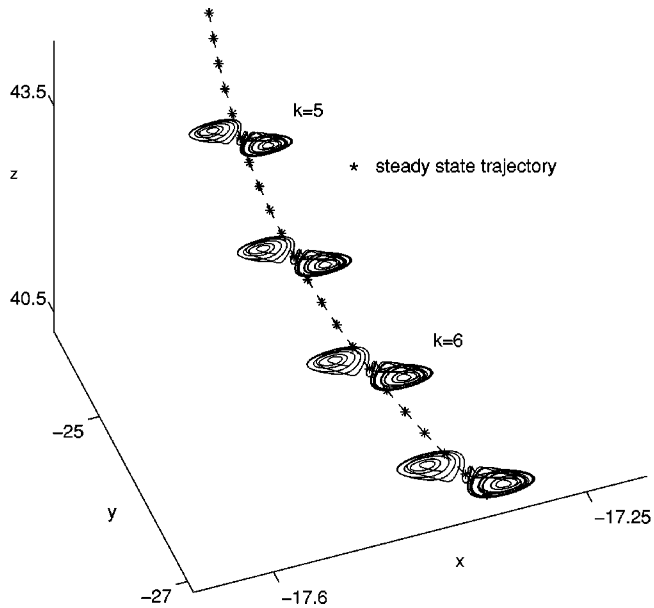

FIG. 10. A Chua system with parameters $\alpha=10$, $c=-0.143$, $\beta=16$ guides a Lorenz system with parameters $\sigma=16$, $\rho=45.92$, $\beta=4$ at four different values of the gain parameter, $k=5$, $k=5.5$, $k=6$, and $k=6.5$. The trajectory in phase space of one of the steady states of the perturbed Lorenz system in that range of values of the gain parameter k is also shown.

As k increases, the family of perturbed autonomous Lorenz systems changes the position and stability of its natural equilibrium points according to the following bifurcation pattern. Initially, for $k=0$, the unperturbed Lorenz system has three unstable equilibrium states. One saddle point at the origin and two symmetric unstable spiral points at a certain distance from it. The origin remains an equilibrium point for the whole range of values of the parameter k . The two spiral points move symmetrically towards zero, as k increases, changing their stability condition to stable spiral equilibrium points at $k=0.77$ and to stable nodes at $k=17.24$. Finally, at $k=19.63$, the two fixed points collapse to zero and disappear, according to a pitchfork bifurcation pattern where the zero unstable fixed point becomes stable. For $k>19.63$, the origin is the only stable equilibrium point. According to this, the perturbed guided system loses its chaotic character from $k=0.77$, and becomes susceptible to being guided by the external Chua system. This situation is reflected in the conditional Lyapunov exponents, the three of them becoming negative from $k=0.77$ generating conditions for the appearance of a regime of phase synchronization together with generalized synchronization.

As a consequence the guided Lorenz system soon starts synchronizing in frequency with the Chua guiding system and at values of k beyond $k=0.77$ the guided system tries to replicate the driver although at the wrong location in phase space. The location in phase space is determined by the location of the equilibria of the perturbed Lorenz family. Figure 10 shows four instances of Lorenz guided systems at values of k, respectively, $k=5$, $k=5.5$, $k=6$, and $k=6.5$. The figure also shows the movement through phase space that the coupling imposes to one of the originally unstable spiral equilibrium points of the guided Lorenz system in that range of values of the gain parameter k.

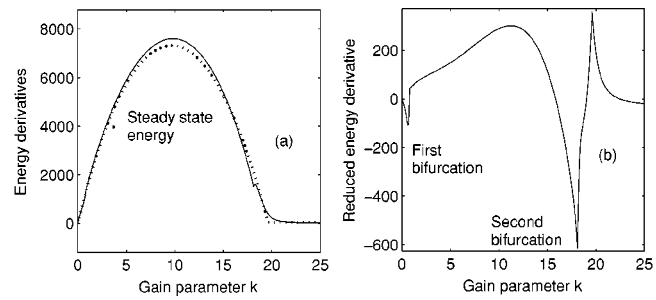

FIG. 11. A Chua system with parameters $\alpha=10$, $c=-0.143$, $\beta=16$ guides a Lorenz system with parameters $\sigma=16$, $\rho=45.92$, $\beta=4$. In part (a) energy derivative corresponding to the theoretical equilibrium regime at each value of k (dots) and actual computed energy derivative (full line). In part (b), difference between both energy derivatives, actual minus theoretical for the steady state.

This behavior implies that the main responsible component of the observed energy exchange pattern is the overall movement in phase space of the oscillatory region of the guided system towards the origin, as the strength of the coupling increases. Superimposed to it, and hidden, remains a flow of energy that could be in some way associated with the cost of the frequency and phase synchronization of both systems. As the energy function and the bifurcation pattern of equilibriums of the guided system are known, the main energy component associated to the location of the guided oscillatory regime can be theoretically calculated. Figure 11(a) shows the derivative of the energy that corresponds to the successive equilibrium points as a function of k. On the same picture the actual computed energy derivative of the guided Lorenz system is also shown. In Fig. 11(b) the difference between both energy derivatives, which we have called reduced energy derivative, is shown. The transition to a stable regime of the intrinsic dynamics of the guided system at $k=0.77$ has a clear effect on this reduced energy derivative. It can be seen in the figure as an initial dissipative transient regime. Once this transient is over, the energy derivative becomes positive for an ample region of values of k. That means that at every value of k within this range the system tries to increase its average energy via a kind of active dissipation. Nevertheless, as the system is confined to a recurrent region of the phase space it must have a constant average energy and, consequently, the coupling device must necessarily absorb the supply of energy. At values of the coupling strength around $k=18$ this situation is reversed. There is a large perturbation, apparently produced by the collapse of the two stable spirals to a stable node at the origin, that makes the guided system become very dissipative. This perturbation relaxes with increasing gains and a less dissipative regime is reached at $k=25$ that continues basically unchanged for any other larger value of the gain parameter.

VI. CONCLUSIONS

In this paper we have presented a method to assign to a chaotic system of known dynamics a function of the phase space variables with the characteristics of an energy. To do so we have used a generalized Hamiltonian formalism with the additional condition that any energy variation along a trajectory of the chaotic system be linked to the divergence of the vector field responsible for the volume contraction of the phase space. In this way, the energy function associated to a system becomes intimately related to its particular structure conveying a real physical meaning. We have assigned energy functions to the Lorenz, Rössler, and Chua families of chaotic systems.

We have used the previously deduced Hamiltonian energies to establish a measure of the cost, in terms of energy, of the maintenance of different degrees of synchronized regimes between these chaotic systems. Our results confirm that the cost of maintaining an identical synchronized regime between identical systems is zero. Nevertheless, for coupling strengths weaker than the required to establish an identical synchronization regime a continuous supply of energy is demanded by the guided system that slumps to zero at the onset of the identical synchronization. We have observed this behavior performing computational experiments of identical synchronization with both, a Chua guided Chua system and a Lorenz guided Lorenz system.

A synchronized regime between systems of different structures never occurs spontaneously at any given value of the gain parameter k. Complete synchronization between nonidentical systems is a limit regime that requires coupling strengths going to infinity. Nevertheless, a theoretically infinite coupling strength does not mean an unlimited provision of energy. We have proved that, in the limit, maintaining an identical synchronized regime between nonidentical systems requires a limited energy per unit time that can be found averaging the dissipation of energy of the guided system along a trajectory of the guiding system. Thus, this limit value of the average energy depends on the particular characteristics of both systems.

This result can be used to evaluate the robustness of the synchronization between supposedly identical systems to parameter mismatch. We have illustrated this point studying the sensitivity of the synchronization energy to parameter mismatch in the case of a Lorenz system guiding another identical Lorenz system.

We have also studied and discussed the flows of energy at different values of the coupling strength for nonidentical synchronization between Chua guided both Rössler and Lorenz systems. We show that the dissipated energy is sensitive to some salient features of the bifurcation pattern of equilibria of the perturbed guided system and it is able to detect some of their transitions to stability. We show that the whole contribution to the dissipated energy can be analyzed in terms of two components. The quantitatively most important component can be associated with the trajectory described in phase space by the stable steady states induced by the coupling in the guided system. The rest of the dissipated energy could be in some way related to the accomplishment of the synchronization in frequency and phase.

Finally, we think that the energy approach developed in this paper can be used to explore some of the well-established phenomena in the synchronization of chaotic systems such as the collapse of the trajectory of the coupled system to some invariant subspaces or the intermittent loss of synchronization in coupled identical systems when the coupling strength is just beyond the synchronization threshold and the system is subjected to small perturbations. The transition to stability of the synchronization manifold is revealed by the transition to negative values of the conditional Lyapunov exponents. Nevertheless, if the Lyapunov exponents are negative and yet there are persistent desynchronization events with synchronization errors reaching levels beyond that of the perturbations, it is because there are invariant sets locally unstable which are able to magnify the perturbation. These regions in phase space might be characterized by positive local Lyapunov exponents. An analysis of the local average energy variation corresponding to the regions in phase space with positive local Lyapunov exponents would show the way the synchronization energy can be sensitive to the influence of the unstable sets of the synchronization manifold.

ACKNOWLEDGMENTS

This work was partly supported by Gipuzkoako Foru Aldundia and the Basque Government.

[1] L.M. Pecora, T.L. Carroll, G.A. Johson, D.J. Mar, and J.F. Heagy Chaos **7**, 520 (1997); U. Parlitz and L. Kocarev, *Handbook of Chaos and Control*, edited by H. G. Shuster, (Wiley-VCH, Berlin, 1999), pp. 271–303; S. Boccaletti, J. Kurths, G. Osipov, D.L. Valladares and C.S. Zhou Phys. Rep. **366**, 1 (2002).

[2] C. Sarasola, F.J. Torrealdea, A. d'Anjou, and M. Graña, Math. Comput. Simul. **58**, 309 (2002); C. Sarasola, F.J. Torrealdea, A. d'Anjou, A. Moujahid, and M. Graña, Physica D **177**, 39 (2003).

[3] L. Kocarev, A. Shang, and L.O. Chua, Int. J. Bifurcation Chaos Appl. Sci. Eng. **3**, 479 (1993).

[4] L.M. Pecora and T.L. Carroll Phys. Rev. Lett. **64**, 821 (1990).

[5] L.M. Pecora, T.L. Carroll, G.A. Johson, D.J. Mar, and J.F. Heagy Chaos **7**, 520 (1997).

[6] V. I. Arnold *Mathematical Methods of Classical Mechanics*, (Springer, New York,1989).

[7] R.I. McLachlan, G.R.W. Quispel, and N. Robidoux Phys. Rev. Lett. **81**, 2399 (1998).

[8] A. Bloch, P.S. Krishnaprasad, J.E. Marsden, and T.S. Ratiu Commun. Math. Phys. **175**, 1 (1996).

[9] H. Sira-Ramirez and C. Cruz-Hernandez, Int. J. Bifurcation Chaos Appl. Sci. Eng. **11**, 1381 (2001).

[10] B. M. J. Maschke, R. Ortega, and A.J. Van Der Shaft, Proceedings of CDC98, Tampa, Florida (unpublished), pp. 3599–3604.

[11] D. Cheng and S. Spurgeon, Int. J. Control **74**, 465 (2001).

[12] A. Hohl, A. Gavrielides, T. Erneux, and V. Kovanis, Phys. Rev. Lett. **78**, 4745 (1997).
[13] P. Tass, M.G. Rosenblum, J. Weule, J. Kurths, A. Pikovsky, J. Volkmann, A. Schnitzler, and H.J. Freund, Phys. Rev. Lett. **81**, 3291 (1998).
[14] M.B. Lotric and A. Stefanovska, Physica A **283**, 451 (2000).
[15] R. Mrowka, A. Patzak, and M. Rosenblum, Int. J. Bifurcation Chaos Appl. Sci. Eng. **10**, 2479 (2000).
[16] W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, Addison-Wesley Series in Physics (Addison-Wesley, Reading, MA, 1962), pp. 2–7; H. Kobe Donald, Am. J. Phys. **54**, 552 (1986).
[17] C. Sarasola, F.J. Torrealdea, A. d'Anjou, A. Moujahid, and M. Graña, Int. J. Bifurcation Chaos Appl. Sci. Eng. **13**, 177 (2003).
[18] L. Kocarev and U. Parlitz, Phys. Rev. Lett. **76**, 1816 (1996).